Revision Exercises (Advanced)

Q1 $3x+4 < x^2-6 < 9-2x$

$3x+4 < x^2-6$ $\qquad\qquad$ $x^2-6 < 9-2x$

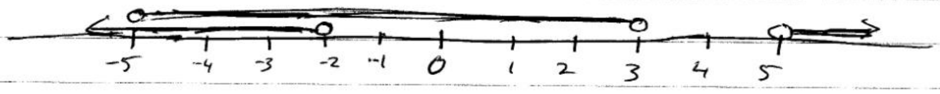
$-x^2+3x+10 < 0$ $\qquad\qquad$ $x^2+2x-15 < 0$

$x^2 - 3x - 10 > 0$ $\qquad\qquad$ $(x-3)(x+5) < 0$

$(x-5)(x+2) > 0$ $\qquad\qquad$ $x < 3 \quad x < -5.$

$x > 5 \quad x > -2$

$-2 > x > 5$ $\qquad\qquad$ $-5 < x < 3$



Solution $-5 < x < -2.$

Q2

$$M = 30 \times 2^{-0.001t}$$

(i) original $\Rightarrow t=0$ $\qquad$ $M = 30 \times 2^0 = 30 \times 1 = 30g$

(ii) $30 \times 2^{-0.001t} = 10$

$2^{-0.001t} = 10/30 = 1/3$

$\log 2^{-0.001t} = \log 1/3$

$-0.001t \log 2 = \log 1/3$

$-0.001t = \dfrac{\log 1/3}{\log 2}$

$-0.001t = -1.58496$

$t = 1584.96$

$t = 1585$ years

Q2 (iii) 1% of $30 = 0.3g$.

$$30 \times 2^{-0.001t} = 0.3$$
$$2^{-0.001t} = \frac{0.3}{30} = 0.01$$
$$\log 2^{-0.001t} = \log 0.01$$
$$-0.001t \log 2 = \log 0.01$$
$$-0.001t = \frac{\log 0.01}{\log 2}$$
$$-0.001t = -6.6439$$
$$t = 6643.86$$
$$t = 6644 \text{ years}.$$

Q3

$$I = I_0 \times 10^{0.1S}$$

(i) $S = 30 \Rightarrow I = I_0 \times 10^{0.1(30)}$
$$I = I_0 \times 10^3$$
$$I = I_0 \times 1000.$$

Theshold $= I_0$ when $S = 30 = 1000 I_0$
$\Rightarrow$ 1000 times loulder

(ii) $S = 28 \Rightarrow I = I_0 \times 10^{0.1(28)} = I_0 \times 10^{2.8} = I_0(630.957)$
$S = 15 \Rightarrow I = I_0 \times 10^{0.1(15)} = I_0 \times 10^{1.5} = I_0(31.623)$

Times loulder $= \frac{\cancel{I_0} 630.957}{\cancel{I_0} 31.623} = 19.95 \approx 20$ times loulder

Q4

$$\log_5 x - 1 = 6 \log_x 5$$

$$\log_5 x - 1 = 6 \frac{\log_5 5}{\log_5 x}$$

$$\log_5 x - 1 = 6 \frac{1}{\log_5 x} \qquad \text{let } y = \log_5 x$$

$y - 1 = 6\left(\frac{1}{y}\right)$ mult by y

$$y^2 - y = 6$$
$$y^2 - y - 6 = 0$$
$$(y-3)(y+2) = 0$$
$$y = 3 \qquad y = -2$$

find x: $y = \log_5 x$

$$3 = \log_5 x \qquad -2 = \log_5 x$$
$$5^3 = x \qquad 5^{-2} = x$$
$$125 = x \qquad \tfrac{1}{25} = x$$

Q5

$$0.7^x \geq 0.3$$
$$\log (0.7)^x \geq \log 0.3$$
$$x \log 0.7 \geq \log 0.3$$
$$x(-0.1549) \geq -0.5229.$$

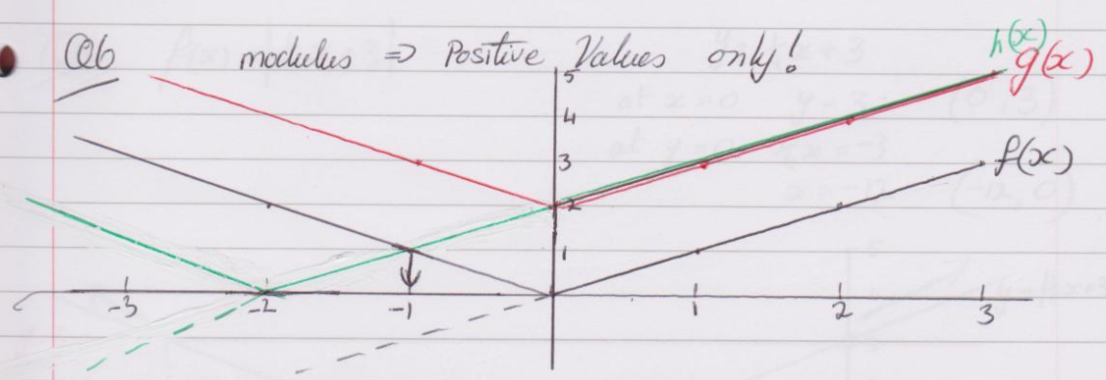
(Note: Cannot bring $\log 0.7$ across the inequality as do not know if it is Pos or Neg)

change Signs $\quad x(0.1549) \leq 0.5229$

$$x \leq \frac{0.5229}{0.1549}$$

$$x \leq 3.3755$$
$$x \leq 3.38.$$

Q6 modulus ⇒ Positive Values only!



(i) $f(x) = |x|$ ⇒ (1, 1) (2, 2) etc

(ii) $g(x) = |x| + 2$ ⇒ (1,3) (2,4) (3,5) etc   y intercept = 2

↳ same as f(x) but lift 2.

(iii) $h(x) = |x+2|$ ⇒ (1, 3) (2, 4)
(−1, 1) (−2, 0) (−3, 1)

(iv) $f(x) \cap h(x)$ at $x = -1$

(v) $g(x) > h(x)$ at $-3 \leq x < 0$

(note: the inequality signs)

Q7 $y = \ln(x-3)$

x	$y = \ln(x-3)$	y
4	ln (1)	0
5	ln(2)	0·7
6	ln(3)	1·1
7	ln(4)	1·4
8	ln 5	1·6

Rem ln zero or neg is undefined!

(4,0) (5,0·7) (6,1·1) (7,1·4) (8,1·6)

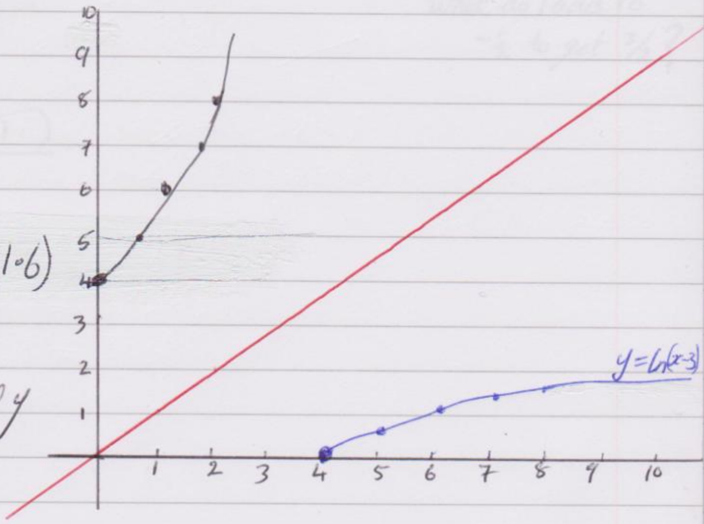


image = the inverse

→ express x in terms of y

$y = \ln_e x - 3$

$e^y = x - 3$

$\boxed{e^y + 3 = x}$

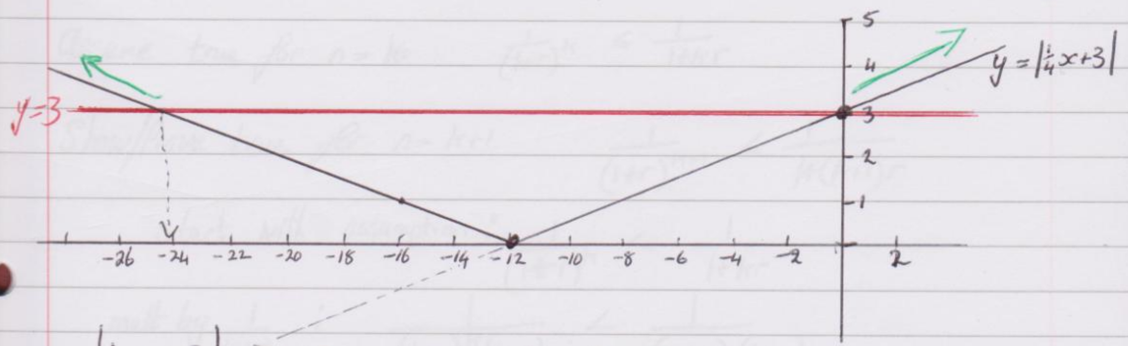
Q8 $f(x) = \left|\frac{1}{4}x + 3\right|$

$y = \frac{1}{4}x + 3$

at $x = 0$ $\quad y = 3 \quad (0, 3)$

at $y = 0$ $\quad \frac{1}{4}x = -3$

$x = -12 \quad (-12, 0)$



$\left|\frac{1}{4}x + 3\right| \geqslant 3 \qquad -24 \geqslant x \geqslant 0$

Q9 $\dfrac{x^{3/2} - x^{-1/2}}{x^{1/2} - x^{-1/2}}$ factorise!

$= \dfrac{\cancel{x^{-1/2}}(x^2 - 1)}{\cancel{x^{-1/2}}(x - 1)}$

$x^{-1/2}(x^{\square}) = x^{3/2}$

what do I add to $-\frac{1}{2}$ to get $\frac{3}{2}$?

$= \dfrac{(x+1)\cancel{(x-1)}}{\cancel{(x-1)}}$

$= x + 1$

Q10 Prove $\frac{1}{(1+r)^n} \leq \frac{1}{1+nr}$ for $r > 0$, $n \in N$.

Show true for $n=1$ $\quad \frac{1}{(1+r)^1} \leq \frac{1}{1+(1)r} \Rightarrow \frac{1}{1+r} = \frac{1}{1+r}$

Assume true for $n=k$ $\quad \frac{1}{(1+r)^k} \leq \frac{1}{1+kr}$

Show/Prove true for $n=k+1$ $\quad \frac{1}{(1+r)^{k+1}} \leq \frac{1}{1+(k+1)r}$

start with assumption: $\frac{1}{(1+r)^k} < \frac{1}{1+kr}$

mult by $\frac{1}{(1+r)}$: $\quad \frac{1}{(1+r)^k(1+r)} < \frac{1}{(1+kr)(1+r)}$

$$\frac{1}{(1+r)^{k+1}} < \frac{1}{1+r+kr+kr^2}$$

$$\frac{1}{(1+r)^{k+1}} < \frac{1}{1+r(k+1)+kr^2} \qquad kr^2 > 0 \text{ for } k \geq 1,\ r > 0$$

$$\Rightarrow \frac{1}{(1+r)^{k+1}} < \frac{1}{1+(k+1)r} \Rightarrow \text{true for } n=k+1$$

Since true for $n=1$, $\Rightarrow$ true for $n=1+1=2$.
Since true for $n=2$ $\Rightarrow$ true for $n=2+1=3$.
$\therefore$ true for all values of n, $n \in N$.

Q11 $\dfrac{4x}{(x+1)^2} \leq 1$ $x \in R$

mult by $(x+1)^2$ $\dfrac{4x}{(x+1)^2} \times (x+1)^2 < 1\,(x+1)^2$

$4x < (x+1)^2$

$4x < x^2 + 2x + 1$

$-x^2 + 2x - 1 < 0$

change signs $x^2 - 2x + 1 > 0$ Note: Change Inequality

$(x-1)(x-1) > 0$

$(x \quad 1)^2 > 0$

anything squared is $> 0 \Rightarrow$ true.

Q12 $(1+2k)x^2 - 10x + (k-2) = 0$

(i) find k such that roots are real.

real roots $\Rightarrow$ $b^2 - 4ac \geq 0$

$(-10)^2 - 4(1+2k)(k-2) \geq 0$

$100 - [4k - 8 + 8k^2 - 16k] \geq 0$

$100 - 4k + 8 - 8k^2 + 16k \geq 0$

$-8k^2 + 12k + 108 \geq 0$

change signs $8k^2 - 12k - 108 \leq 0$ change Inequality.

$(\div 4)$ $2k^2 - 3k - 27 \leq 0$

$(2k - 9)(k + 3) \leq 0$

$k \leq \frac{9}{2}$ $k \leq -3$

$-3 \leq k \leq \frac{9}{2}$

Q12 (ii) Value of k for which roots sum > 5.

Eqn canbe written x^2 sum$x$ + product $= 0$

$(1+2k)x^2 - 10x + (k-2) = 0$

$\div (1+2k)$ $\quad x^2 - \frac{10}{1+2k} + \frac{k-2}{1+2k} = 0$

$\Rightarrow$ Sum of roots $= \frac{10}{1+2k}$

$\frac{10}{1+2k} > 5$

mult by $(1+2k)^2$: $\quad \frac{10}{1+2k}(1+2k)^2 > 5(1+2k)^2$

$10(1+2k) > 5(1+4k+4k^2)$

$10+\cancel{20k} > 5+\cancel{20k}+20k^2$

$-20k^2+5 > 0$

change Signs $\quad 20k^2 - 5 < 0$

$(\div 5)$ $\quad 4k^2 - 1 < 0$

$(2k+1)(2k-1) < 0$

$k < -\frac{1}{2} \quad k < \frac{1}{2}$

$\boxed{-\frac{1}{2} < k < \frac{1}{2}}$

Q13 $1 + (2)2 + (3)2^2 + (4)2^3 + \cdots (n)2^{n-1} = (n-1)2^n + 1$

Show true for $n=1$ $\quad 1 = (1-1)2^1 + 1, \quad 1 = (0)2^1+1, \ 1=1$, True.

Assume true for $n=k$

$$1 + (2)2 + (3)(2)^2 + (4)2^3 + \cdots (k)2^{k-1} = (k-1)2^k + 1$$

Prove true for $n = k+1$

ie. ~~Prove~~ To Show $((k+1)-1)2^{k+1} + 1$.

start with ~~original~~ assumed and add $(k+1)2^{(k+1)-1}$ to both sides.

$$1 + (2)2 + (3)2^2 + 4(2)^3 + \cdots (k)2^{k-1} + (k+1)2^{(k+1)-1} = (k-1)2^k + 1 + (k+1)2^{(k+1)-1}$$

$$= (k-1)2^k + 1 + (k+1)2^k$$

$$= (k)2^k - \cancel{2^k} + 1 + (k)2^k + \cancel{2^k}$$

$$= (k)2^k + (k)2^k + 1$$

$$= (2)(k)2^k + 1$$

$$= 2^{k+1}(k) + 1$$

$$= 2^{k+1}(k+1-1) + 1$$

$\Rightarrow$ True for $n = k+1$.

Since true for $n=1$, $\Rightarrow$ true for $n = 1+1 = 2$

Since true for $n=2$ $\Rightarrow$ true for $n = 2+1 = 3$ etc

$\therefore$ true for all values of n.

Q14 $U_n = (n-20)2^n$

$U_{n+1} \Rightarrow ((n+1)-20)2^{n+1} = (n-19)2^{n+1}$

$U_{n+2} \Rightarrow ((n+2)-20)2^{n+2} = (n-18)2^{n+2}$

Verify: $U_{n+2} - 4U_{n+1} + 4U_n = 0$

$(n-18)2^{n+2} - 4(n-19)2^{n+1} + 4(n-20)2^n = 0$

$(n-18)2^n(2)^2 - 4(n-19)(2^n)(2) + 4(n-20)2^n = 0$

$2^n[4(n-18) - 4(n-19)(2) + 4(n-20)] = 0$

$2^n[\cancel{4n} - 72 - \cancel{8n} + 152 + \cancel{4n} - 80] = 0$

$2^n(0) = 0$

$0 = 0$ True.

Q15 $2\log y = \log 2 + \log x$ and $2^y = 4^x$

$\log y^2 = \log 2x$ $\qquad 2^y = 2^{2x}$

$y^2 = 2x \leftarrow$ sub in $y = 2x$

$y^2 = y$

$y^2 - y = 0$

$y(y-1) = 0$

$y = 0$ $\qquad \boxed{y=1}$

(not Valid) $\log(0)$

$y = 1 \Rightarrow y = 2x$

$1 = 2x$

$\boxed{1/2 = x}$

Q16 $P = 40{,}000(1{\cdot}03)^n$

(i) It is an exponential function

(ii) $P = 40{,}000(1{\cdot}03)^{12} = 57030.$

(iii) Initial $\Rightarrow n=0.$ $P = 40{,}000(1{\cdot}03)^0 = 40{,}000.$

(iv) Double $\Rightarrow P = 80{,}000.$

$$80{,}000 = 40000(1{\cdot}03)^n$$
$$2 = (1{\cdot}03)^n$$
$$\log 2 = \log(1{\cdot}03)^n$$
$$\log 2 = n\log(1{\cdot}03)$$
$$\frac{\log 2}{\log 1{\cdot}03} = n$$
$$23{\cdot}45 = n$$
$$23{\cdot}5 \text{ yrs} = n$$

Q17 (i) $P = Ae^{kt}$ where $A = 8000$, $P = 15{,}000$, $t = 8$
must find k.

$15{,}000 = 8000e^{8k}$

$\frac{15{,}000}{8000} = e^{8k}$

$1.875 = e^{8k}$

$\ln 1.875 = \ln e^{8k}$

$\ln 1.875 = 8k \ln e$

$\frac{\ln 1.875}{\ln e} = 8k$

$0.6286 = 8k$

$0.078576 = k$

$\Rightarrow P = Ae^{kt}$, $A = 8000$, $P = 15{,}000$, $t = 8$, $k = 0.078576$

(ii) End of 2009 $\Rightarrow t = 10$

$P = 8000e^{(0.078576)(10)}$

$P = 8000e^{0.78576}$

$P = 17552.59$

$P = 17553$

(iii) 2007 $P = 15{,}000$ $\Rightarrow$ Double $= 30{,}000$

$30{,}000 = 8000e^{0.078576(t)}$

$3.75 = e^{0.078576t}$

$\ln 3.75 = 0.078576t \underbrace{\boxed{\ln(e)}}_{=1}$

$\frac{1.32175584}{0.078576} = t$

$16.82 = t$

17yrs $= t$ 2000 plus 17 yrs $\Rightarrow$ end of 2016.